1. Solve the differential equation
   $$y''(x) - 4\lambda y'(x) + 4\lambda^2 y(x) = 0, \qquad y'(1) = 0, \quad y(2) + 2y'(2) = 0$$
   (10%)

2. Solve the differential equation
   $$(ax+b)^2\, y''(x) - 2(ax+b)\,y'(x) - 6ay(x) = 3ax + 2b$$
   in terms of $(ax+b)$, where a and b are constants. (10%)

3. Solve the following equation using the Fourier Transform: An engineer, John, tried to predict the vibration of a dynamic system. It was found that the response of the dynamic system can be approximately represented by the following equation:

   $$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t)$$

   where x is the vibration magnitude of the system; m, c, and k are constants, and f(t), the external force, is a continuous function of time. John measured the external force every τ seconds. The measured data are f_1, f_2, ..f_N. And it was found that $f_i = f_{i+100}$ for any integer i. What would be the predicted vibration of the system when $x(0) = 0$, $x(0) = 1$?

   (18%)

4. A system of linear algebraic equations can be expressed as
   $$\mathbf{Ax} = \mathbf{b}$$
   where **A** is a 3 × 3 matrix and **b** is 3 × 1 vector
   $$\mathbf{A} = \begin{bmatrix} \alpha & \alpha & 3 \\ 0 & \alpha & \beta \\ \alpha & 0 & 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ \beta \end{bmatrix}$$
   Determine for what fixed values of α and β (if any) the system possesses the following:
   a) A unique solution. (2%)
   b) A one-parameter solution. (2%)
   c) A two-parameter solution. (2%)
   d) No solution. (2%)

5. *Consider the so-called generalized eigenvalue problem*

$$\mathbf{Ax} = \lambda\mathbf{Bx}$$

Show that the eigenvalues of $\mathbf{Ax} = \lambda\mathbf{Bx}$ are real numbers if **A** and **B** are real and symmetric and at least one of **A** and **B** is *positive definite*. (8%)

6. Show that the Laplace transform of ln (t) is

$$\mathcal{L}\,[\ln(t)] = \frac{\Gamma'(1) - \ln s}{s}$$

where the gamma function is defined as

$$\Gamma(r) = \int_0^\infty u^{r-1} e^{-u}\, du \qquad (10\%)$$

7. By Newton's law of gravitation, the force of attraction between two particles is

$$\vec{p} = -\frac{a}{r^3} = -a\left(\frac{x - x_0}{r^3}\,\vec{i} + \frac{y - y_0}{r^3}\,\vec{j} + \frac{z - z_0}{r^3}\,\vec{k}\right)$$

where a is a constant and r is the distance between the particles.

a) Find $\nabla \times \vec{p}$. (2%)

b) If we define a scalar function $f(x, y, z) = \dfrac{a}{r}$, what is the relation between $\vec{p}$ and f? (2%)

c) Show that f satisfies the Laplace's equation. (2%)

d) Calculate $\oint_C \vec{p} \bullet d\vec{r}$, where C is the loop defined as $x^2 + 4y^2 = 4$, z = 0. Also give the physical interpretation of your result. (4%)

8. Evaluate

$$\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{(z^2 + 1)}\,dz$$

if t > 0 and C is the circle $|z| = 3$. (8%)

9. Consider the following transient one-dimensional partial differential equations respectively for a string and circular region

$$\frac{\partial T}{\partial t} = c^2 \frac{\partial^2 T}{\partial x^2}, \quad T(x,0) = 1, \quad T(0,t) = T(L,t) = 0.$$

$$\frac{\partial T}{\partial t} = c^2 \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} \right), \quad T(r,0) = 1, \quad T(R,t) = 0.$$

where L is the string length, R is the radius of the circular region.

a) Solve the two problems respectively. For the series of eigenfunction expansion you obtain, you don't need to determine the coefficients for the terms, but describe how you will determine them. (14%)

b) Schematically plot the variation for T(x) or T(r) at some typical instants of time. (4%)