類組：物理類　科目：應用數學(2001)　　※請在答案卷內作答

1. (a) Write down the Cauchy integral formula and prove it. (15 points)

(b) Apply (a) to calculate $\oint_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)(z-2)} dz$ (10 points)

(c) If f(z) is analytical inside and on a closed contour C, calculate

$$\oint_C \frac{f'(z)}{f(z)} dz$$ (10 points)

2. (a) A square matrix A is "orthogonally diagonalizable" if there exists an orthogonal matrix C such that $D = C^T AC$ is a diagonal matrix. Prove that A is orthogonally diagonalizable if and only if it is a symmetric matrix. (10 points)

(b) Orthogonally diagonalize $A = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$ and what are the matrices C and D? (10 points)

3. Find the Fourier transform of the function

$$f(t) = \begin{cases} 0, & t < 0 \\ e^{-t/\tau} \sin \omega_0 t, & t \geq 0 \end{cases}$$ (10 points)

4. The vector field $\mathbf{F}$ is given by

$$\mathbf{F} = (3x^2yz + y^3z + xe^{-x})\mathbf{i} + (3xy^2z + x^3z + ye^x)\mathbf{j} + (x^3y + y^3x + xy^2z^2)\mathbf{k}$$

Calculate the value of the line integral $\int_L \boldsymbol{F} \cdot \boldsymbol{dr}$ where L is the 3D closed contour OABCDEO defined by the successive vertices (0, 0, 0), (1, 0, 0), (1, 0, 1), (1, 1, 1), (1, 1, 0), (0, 1, 0), (0, 0, 0). (10 points)

5. Solve

$$\frac{\partial^2 u}{\partial^2 x} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial^2 y} = 0,$$

subject to the boundary condition $u(0, y) = 0$ and $u(x, 1) = x^2$. (10 points)

6. The Laplace transform of $f(t)$ is defined as

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t) dt = F(s)$$

Show that

(a) $$\mathcal{L}\left[\frac{d^2 f(t)}{dt^2}\right] = -\left.\frac{df}{dt}\right|_0 + s(sF(s) - f(0))$$

(b) $$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n F(s)}{ds^n}, \text{ for n = 1, 2, 3,...}$$

(15 points)