國立清華大學命題紙

八十六學年度轉學生入學考試

科目 微積分(一般) 共 1 頁第 1 頁 *請在試卷【答案卷】內作答

I、填充題（共八題，每題9分，請將答案依甲、乙、丙...次序作答，不需演算過程）

1. If $a_n = (1 + \frac{2}{n} + \frac{3}{n^2})^n$, then $\lim_{n\to\infty} a_n =$ 甲 .

2. $\lim_{n\to\infty} \sum_{k=1}^{n} \ln \sqrt[n]{1 + \frac{k}{n}} =$ 乙 .

3. If a, b will maximalize the value of the integral $\int_a^b (4 - x^2)dx$, then $b - a =$ 丙 .

4. Let f be a twice differentiable function and let $u(x) = f(x^2 - 1)$. If $f'(0) = 1$, $f''(0) = 2$, then $u''(1) =$ 丁 .

5. If $\frac{dx}{dt} = (1 - x)(2 - 3x)$, then $\lim_{t\to\infty} x(t) =$ 戊 .

6. The tangent plane to $x^2 + 2y^2 + z^2 = 7$ at $(1, 1, 2)$ is 己 .

7. $\int_0^1 \frac{x^2 + 1}{x^4 + 1} dx =$ 庚 .

8. Denote by D the region $D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$. The value of the double integral $\iint_D |x + y| dA$ is 辛 .

II、計算與証明(必須寫出演算証明過程)

1 (14%)

(a) Prove that the equation $x^3 + x^2 + x = a$ has only one real solution for every real number a.

(b) The sequence $(x_n)_{n=1}^{\infty}$ of real numbers satisfying the following relations

$$x_1 > 0, x_{n+1}^3 + x_{n+1}^2 + x_{n+1} = x_n, \ n \geq 1.$$

Prove that the sequence $(x_n)_{n=1}^{\infty}$ is convergent and find its limit.

2 (14%)

If $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x + y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$ and $f(x) = 1 + xg(x)$ where $\lim_{x\to 0} g(x) = 1$. Show that f is differentiable, and find $f(x)$.